HUREWICZ THEOREM FOR ASSOUAD-NAGATA DIMENSION

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ABSTRACT. Given a function $f\colon X \to Y$ of metric spaces, its *asymptotic dimension* $\mathrm{asdim}(f)$ is the supremum of $\mathrm{asdim}(A)$ such that $A \subset X$ and $\mathrm{asdim}(f(A)) = 0$. Our main result is

Theorem 0.1. $asdim(X) \leq asdim(f) + asdim(Y)$ *for any large scale uniform function* $f\colon X \to Y$.

0.1 generalizes a result of Bell and Dranishnikov [3] in which f is Lipschitz and X is geodesic. We provide analogs of 0.1 for Assouad-Nagata dimension $\dim_{AN}$ and asymptotic Assouad-Nagata dimension asdim_{AN}. In case of linearly controlled asymptotic dimension l-asdim we provide counterexamples to three questions of Dranishnikov [14].

As an application of analogs of 0.1 we prove

Theorem 0.2. *If* $1 \to K \to G \to H \to 1$ *is an exact sequence of groups and* G *is finitely generated, then*

$$asdim_{AN}(G, d_G) \leq asdim_{AN}(K, d_G|K) + asdim_{AN}(H, d_H)$$

for any word metrics metrics d_G *on* G *and* d_H *on* H.

0.2 extends a result of Bell and Dranishnikov [3] for asymptotic dimension.

CONTENTS

1. Introduction

The well-known Hurewicz Theorem for maps (also known as Dimension-Lowering Theorem, see [18], Theorem 1.12.4 on p.109) says $\dim(X) \leq \dim(f) + \dim(Y)$ if $f \colon X \to Y$ is a closed map of separable metric spaces and $\dim(f)$ is defined as the supremum of $\dim(f^{-1}(y))$, $y \in Y$. Bell and Dranishnikov [3] proved a variant of Hurewicz Theorem for asymptotic dimension without defining the asymptotic dimension of a function. However, Theorem 1 of [3] may be restated as $\mathrm{asdim}(X) \leq \mathrm{asdim}(f) + \mathrm{asdim}(Y)$, where $\mathrm{asdim}(f)$ is the smallest integer n such that $\mathrm{asdim}(f^{-1}(B_R(y))) \leq n$ uniformly for all $R > 0$. As an application it is shown in [3] that $\mathrm{asdim}(G) \leq \mathrm{asdim}(K) + \mathrm{asdim}(H)$ for any exact sequence $1 \to K \to G \to H \to 1$ of finitely generated groups. That inequality was extended subsequently by Dranishnikov and Smith [16] to all countable groups.

The purpose of this paper is to generalize Hurewicz Theorem to variants of asymptotic dimension: asymptotic Assouad-Nagata dimension and Assouad-Nagata dimension. In the process we produce a much simpler proof than that in [3] and a stronger result: the function is only assumed to be large scale uniform instead of Lipschitz, and the domain is not required to be geodesic.

One of the main tools is Kolmogorov's idea used in his solution to Hilbert's 13th Problem. In dimension theory it is known as Ostrand Theorem. Another tool is reformulating Gromov's [19] definition of asymptotic dimension in terms of r-components of spaces. That leads to a definition of asymptotic dimension of a function in terms of double-parameter components, a concept well-suited for Kolmogorov Trick.

2. Ostrand theorem for asymptotic dimension

The aim of this section is to prove a variant of Ostrand's Theorem for large scale dimensions. As an application we present a simple proof of the Logarithmic Law for large scale dimensions.

Intuitively, a metric space is of dimension 0 at scale r if it can be represented as a collection of r-disjoint and uniformly bounded subsets. The following definition of Gromov [19] defines asymptotic dimension

at most n in terms of being represented, for each $r > 0$, as the union of $n+1$ sets of dimension 0 at scale r. That corresponds to the well-known property of topologically n-dimensional spaces.

Definition 2.1. A metric space X is said to be of *asymptotic dimension* at most n (notation: $\mathrm{asdim}(X) \leq n$) if there is a function $D_X \colon R_+ \to R_+$ such that for all $r > 0$ there is a cover $\mathcal{U} = \bigcup_{i=1}^{n+1} \mathcal{U}_i$ of X so that each $\mathcal{U}_i$ is r-disjoint (that means $d(a,b) \geq r$ for any two points a and b belonging to different elements of $\mathcal{U}_i$) and the diameter of elements of $\mathcal{U}$ is bounded by $D_X(r)$.

We refer to the function D_X as *an n-dimensional control function* for X. When discussing variants of asymptotic dimension it is convenient to allow D_X to assume infinity as its value at some range of r.

Definition 2.2. A metric space X is said to be of *Assouad-Nagata dimension* (see [20] and [7]) at most n (notation: $\dim_{AN}(X) \leq n$) if it has an n-dimensional control function D_X that is a dilation ($D_X(r) = c \cdot r$ for some $c > 0$).

A metric space X is said to be of *asymptotic Assouad-Nagata dimension* at most n (notation: $\mathrm{asdim}_{AN}(X) \leq n$) if it has an n-dimensional control function D_X that is linear ($D_X(r) = c \cdot r + b$ for some $b, c \geq 0$).

A metric space X is said to be of *linearly controlled asymptotic dimension* at most n (Dranishnikov [14], notation: l-$\mathrm{asdim}(X) \leq n$) if it has an n-dimensional control function D_X satisfying $D_X(r) = c \cdot r$ for some $c > 0$ and for all r belonging to some unbounded subset of R_+. Strictly speaking, the original definition of Dranishnikov [14] is formulated in terms of Lebesque numbers. However, just as in the case of asymptotic dimension, it is equivalent to our definition.

A metric space X is said to be of *microscopic Assouad-Nagata dimension* at most n if it has an n-dimensional control function $D_X \colon R_+ \to R_+ \cup \infty$ that is a dilation near 0: $D_X(r) = c \cdot r$ for some $c > 0$ and all r smaller than some positive number M, $D_X(r) = \infty$ for all $r \geq M$.

Definition 2.3. Given a metric space X and $k \geq n+1 \geq 1$ an (n,k)-*dimensional control function* for X is a function $D_X \colon R_+ \to R_+$ such that for any $r > 0$ there is a family $\mathcal{U} = \bigcup_{i=1}^{k} \mathcal{U}_i$ satisfying the following conditions:

(1) each $\mathcal{U}_i$ is r-disjoint,
(2) each $\mathcal{U}_i$ is $D_X(r)$-bounded,

(3) each element $x \in X$ belongs to at least $k - n$ elements of $\mathcal{U}$ (equivalently, $\bigcup_{i \in T} \mathcal{U}_i$ is a cover of X for every subset T of $\{1, \ldots, k\}$ consisting of $n + 1$ elements).

The following result is an adaptation of a theorem by Ostrand [25].

Theorem 2.4. *If $D_X^{(n+1)}$ is an n-dimensional control function of X and one defines a sequence of functions $\{D_X^{(i)}\}_{i \geq n+1}$ inductively by $D_X^{(i+1)}(r) = D_X^{(i)}(3r) + 2r$ for all $i \geq n + 1$, then each $D_X^{(k)}$ is an (n, k)-dimensional control function of X for all $k \geq n + 1$.*

Proof. The proof is by induction on k. The case of $k = n + 1$ is obvious. Suppose the result holds for some $k \geq n+1$. Let $\mathcal{U} = \bigcup_{i=1}^{k} \mathcal{U}_i$ be a family such that each $\mathcal{U}_i$ is $3r$-disjoint, each $\mathcal{U}_i$ is $D_X^{(k)}(3r)$-bounded, and each element $x \in X$ belongs to at least $k - n$ elements of $\mathcal{U}$. Define $\mathcal{U}_i'$ to be the r-neighborhoods of elements of $\mathcal{U}_i$ for $i \leq k$. Notice elements of $\mathcal{U}_i'$ are $(D_X^{(k)}(3r) + 2r)$-bounded and are r-disjoint. Define $\mathcal{U}_{k+1}'$ as the collection of all sets of the form $\bigcap_{s \in S} A_s \setminus \bigcup_{i \notin S} \mathcal{U}_i'$, where S is a subset of $\{1, \ldots, k\}$ consisting of exactly $k - n$ elements and $A_s \in \mathcal{U}_s$.

Notice that any element of $\mathcal{U}_{k+1}'$ is contained in a single element of some $\mathcal{U}_j$. Thus elements of each $\mathcal{U}_i'$ are $(D_f^{(k)}(3r) + 2r)$-bounded.

Given two different sets $A = \bigcap_{s \in S} A_s \setminus \bigcup_{i \notin S} \mathcal{U}_i'$, where S is a subset of $\{1, \ldots, k\}$ consisting of exactly $k - n$ elements and $A_s \in \mathcal{U}_s$, and $B = \bigcap_{t \in T} B_t \setminus \bigcup_{i \notin T} \mathcal{U}_i'$, where T is a subset of $\{1, \ldots, k\}$ consisting of exactly $k - n$ elements and $B_t \in \mathcal{U}_t$, we need to show A and B are r-disjoint. It is clearly so if $S = T$, so assume $S \neq T$. If $a \in A$, $b \in B$, and $d(a, b) < r$, then there is $s \in S \setminus T$ such that $a \in A_s$ prompting $b \in U \in \mathcal{U}_s'$, a contradiction.

Suppose $x \in X$ belongs exactly to $k - n$ sets $\bigcup \mathcal{U}_i'$, $i \leq k$, and let $S = \{i \leq k \mid x \in \bigcup \mathcal{U}_i'\}$. If $x \notin \bigcup \mathcal{U}_{k+1}'$, then x must belong to $\bigcup \mathcal{U}_j'$ for some $j \notin S$, a contradiction. Thus each $x \in X$ belongs to at least $k + 1 - n$ elements of $\{\bigcup \mathcal{U}_i'\}_{i=1}^{k+1}$. ■

The product theorem for asymptotic dimension was proved in [15] using maps to polyhedra. The product theorem for Nagata dimension was proved in [20] using Lipschitz maps to polyhedra. Below we use Theorem 2.4 to give a simplified proof for all dimension theories. Note the metric on $X \times Y$ is the sum of corresponding metrics on X and Y.

Theorem 2.5. *If X and Y are metric spaces, then*

$$D(X \times Y) \leq D(X) + D(Y),$$

where D stands for any of the following dimension theories: asymptotic dimension, asymptotic Assouad-Nagata dimension, Assouad-Nagata dimension, or microscopic Assouad-Nagata dimension.

Proof. Let $D(X) = m$, $D(Y) = n$ and let $k = m+n+1$. Pick (m, k)-dimension control function D_X of X and (n, k)-dimensional control function D_Y of Y, both of the correct type (arbitrary, linear, dilation, or a dilation near 0). There are families $\{\mathcal{U}_i\}_{i=1}^{k}$ in X and $\{\mathcal{V}_i\}_{i=1}^{k}$ in Y that are r-disjoint and bounded by $D_X(r)$ and $D_Y(r)$ respectively, that cover X and Y at least $k - m$ times and $k - n$ times respectively. Then the family $\{\mathcal{U}_i \times \mathcal{V}_i\}_{i=1}^{k}$ covers $X \times Y$, as for any point (x, y), x is contained in sets from at least $k - m = n+1$ families from $\{\mathcal{U}_i\}_{i=1}^{k}$ and y is contained in sets from at least $k - n = m+1$ families from $\{\mathcal{V}_i\}_{i=1}^{k}$, so there is at least one index j such that x is covered by $\mathcal{U}_j$ and y is covered by $\mathcal{V}_j$. The family $\{\mathcal{U}_i \times \mathcal{V}_i\}_{i=1}^{k}$ is r-disjoint and is bounded by $D_X(r) + D_Y(r)$. ∎

3. Components, dimension, and coarseness

In this section we replace the language of r-disjoint families by the language of r-components. This language has the advantage of being portable to functions. It also allows for simple proofs of known results 3.11-3.12.

Definition 3.1. Let $f \colon X \to Y$ be a function of metric spaces, A is a subset of X, and r_X, r_Y are two positive numbers.

A is (r_X, r_Y)-*bounded* if for any points $x, x' \in A$ we have

$$d_X(x, x') \leq r_X \quad \text{and} \quad d_Y(f(x), f(x')) \leq r_Y.$$

An (r_X, r_Y)-*chain* in A is a sequence of points $x_1, \dots, x_k$ in A such that for every $i < k$ the set $\{x_i, x_{i+1}\}$ is (r_X, r_Y)-bounded.

A is (r_X, r_Y)-*connected* if for any points $x, x' \in A$ can be connected in A by an (r_X, r_Y)-chain.

Notice that any subset A of X is a union of its (r_X, r_Y)-*components* (the maximal (r_X, r_Y)-connected subsets of A).

Definition 3.2. Let $f \colon X \to Y$ be a function of metric spaces. f is called *large scale uniform* if there is function $c_f \colon R_+ \to R_+$ such that $d_X(x, y) \leq r$ implies $d_Y(f(x), f(y)) \leq c_f(r)$. The function c_f will be called *a coarseness control function* of f.

Notice f is Lipschitz if and only if it has a coarseness control function that is a dilation. f is *asymptotically Lipschitz* if and only if it has a coarseness control function that is linear.

The following Lemma describes a useful case in which double parameter components coincide with single parameter components. The proof of this Lemma is an easy exercise.

Lemma 3.3. *Let $f\colon X \to Y$ be a function of metric spaces. $c_f\colon R_+ \to R_+$ is a coarseness control function of f if and only if for any subset A of X its $(r, c_f(r))$-components coincide with its r-components.*

The following Lemma describes the way we construct a subset of X with all (r_X, r_Y)-components being (R_X, R_Y)-bounded. The proof of this Lemma is an easy exercise.

Lemma 3.4. *Let $f\colon X \to Y$ be a function of metric spaces, B be a subset of X, and A be a subset of Y. If all r_X-components of B are R_X-bounded and all r_Y-components of A are R_Y-bounded then all (r_X, r_Y)-components of the set $B \cap f^{-1}(A)$ are (R_X, R_Y)-bounded.*

Lemma 3.5. *Let $f\colon X \to Y$ be a function of metric spaces and A, B be subsets of X. Suppose that all (r_X^A, r_Y^A)-components of A are (R_X^A, R_Y^A)-bounded and all (r_X^B, r_Y^B)-components of B are (R_X^B, R_Y^B)-bounded. If $R_X^B + 2r_X^B < r_X^A$ and $R_Y^B + 2r_Y^B < r_Y^A$ then all (r_X^B, r_Y^B)-components of $A \cup B$ are $(R_X^A + 2r_X^A, R_Y^A + 2r_Y^A)$-bounded.*

Proof. Let $x = x_1, x_2, \ldots, x_n = x'$ form an (r_X^B, r_Y^B)-chain in $A \cup B$.

Notice that, if for some indices $i < j$ we have $x_k \in B$ for all $i < k < j$, then x_{i+1} and x_{j-1} are in one (r_X^B, r_Y^B)-component of B and therefore $d_X(x_{i+1}, x_{j-1}) \leq R_X^B$ and $d_Y(f(x_{i+1}), f(x_{j-1})) \leq R_Y^B$.

If $x_i, x_j \in A$ and $x_k \in B$ for all $i < k < j$, then

$$d_X(x_i, x_j) \leq d_X(x_i, x_{i+1}) + d_X(x_{i+1}, x_{j-1}) + d_X(x_{j-1}, x_j) \leq r_X^B + R_X^B + r_X^B < r_X^A$$

and, similarly, $d_Y(f(x_i), f(x_j)) \leq r_Y^B + R_Y^B + r_Y^B < r_Y^A$. Thus the points x_i, x_j belong to the same (r_X^A, r_Y^A)-component of A. This implies that all points in the chain $x = x_1, x_2, \ldots, x_n = x'$ belonging to A are in one (r_X^A, r_Y^A)-component of A.

Now let x_s be the first point in the chain belonging to A and x_t be the last point in the chain belonging to A. Then

$$d_X(x, x') \leq$$

$$d_X(x_1, x_{s-1}) + d_X(x_{s-1}, x_s) + d_X(x_s, x_t) + d_X(x_t, x_{t+1}) + d_X(x_{t+1}, x_n) \leq$$
$$\leq R_X^B + r_X^B + R_X^A + r_X^B + R_X^B < R_X^A + 2r_X^A.$$

Similarly, $d_Y(f(x), f(x')) \leq R_Y^B + r_Y^B + R_Y^A + r_Y^B + R_Y^B < R_Y^A + 2r_Y^A.$ $\square$

Corollary 3.6. *Let $f\colon X \to Y$ be a function of metric spaces and $\{B_i\}_{i=1}^n$ be subsets of X. Suppose that for every i all $(r_X^{(i)}, r_Y^{(i)})$-components of B_i are $(R_X^{(i)}, R_Y^{(i)})$-bounded. If, for every $i < n$, $R_X^{(i+1)} + 2r_X^{(i+1)} < r_X^{(i)}$ and $R_Y^{(i+1)} + 2r_Y^{(i+1)} < r_Y^{(i)}$, then all $(r_X^{(n)}, r_Y^{(n)})$-components of $\bigcup_{i=1}^n B_i$ are $(R_X^{(1)} + 2r_X^{(1)}, R_Y^{(1)} + 2r_Y^{(1)})$-bounded.*

Proof. By induction using Lemma 3.5. $\qquad\square$

Proposition 3.7. *Suppose A is a subset of a metric space X, $m \geq 0$, and $R > 0$. If D_A is an m-dimensional control function of A, then $D_B(x) := D_A(x + 2R) + 2R$ is an m-dimensional control function of the R-neighborhood $B = B(A, R)$ of A.*

Proof. Given $r > 0$ express A as $\bigcup_{i=1}^{m+1} A_i$ such that $(r + 2R)$-components of A_i are $D_A(r + 2R)$-bounded. Given an r-component of $B_i := B(A_i, R)$, each point in that component is R-close to a single $(r + 2R)$-component of A_i. Therefore r-components of B_i are $(D_A(r + 2R) + 2R)$-bounded. $\qquad\blacksquare$

Definition 3.8. Given a metric space X and $r > 0$ the *r-scale dimension* r-dim(X) is the smallest integer $n \geq 0$ such that X can be expressed as $X = \bigcup_{i=1}^{n+1} X_i$ and r-components of each X_i are uniformly bounded.

Notice asdim(X) is the smallest integer n such that r-dim$(X) \leq n$ for all $r > 0$. Also, asdim(X) is the smallest integer such that for each $r > 0$ the space X can be expressed as $X = \bigcup_{i=1}^{n+1} X_i$ so that r-dim$(X_i) \leq 0$ for each $i \leq n + 1$.

Corollary 3.9. *Suppose X is a metric space. If, for every $r > 0$, there is a subspace X_r of X such that $asdim(X_r) \leq n$ and $r\text{-}dim(X \setminus X_r) \leq n$, then $asdim(X) \leq n$.*

Proof. Express $X \setminus X_r$ as $\bigcup_{i=1}^{n+1} A_i$ such that r-components of A_i are R-bounded. Express X_r as $\bigcup_{i=1}^{n+1} B_i$ so that $(R + 2r)$-components of B_i are M-bounded. By Lemma 3.5, r-components of $A_i \cup B_i$ are $(M + 2R + 4r)$-bounded. $\qquad\blacksquare$

Corollary 3.10. *Suppose G is a group with a left-invariant metric and G_r is the subgroup of G generated by $B(1_G, r)$. If, for every $r > 0$, $asdim(G_r) \leq n$, then $asdim(G) \leq n$.*

Proof. Consider $g_s \in G$, $s \in S$, so that $[g_s]$ enumerate $G/G_r \setminus [G_r]$. Notice $Y = \bigcup\limits_{s \in S} g_s \cdot G_r$ is of r-dim$(Y) \leq$ asdim(G_r) and $Y \cup G_r = G$. ∎

Corollary 3.11 (Dranishnikov-Smith [16]). *If G is a group with a proper left-invariant metric, then $asdim(G)$ is the supremum of $asdim(H)$ over all finitely generated subgroups H of G.*

Proof. Since balls $B(1_G, r)$ are finite, the groups G_r in 3.10 are finitely generated. ∎

Corollary 3.12 (Bell-Dranishnikov [3]). *Suppose X is a metric space and $\{X_s\}_{s \in S}$ is a family of subsets of X such that $X = \bigcup\limits_{s \in S} X_s$ and there is a single n-dimensional control function D for all X_s. If, for every $r > 0$, there is a subspace X_r of X such that $asdim(X_r) \leq n$ and the family $\{X_s \setminus X_r\}_{s \in S}$ is r-disjoint, then $asdim(X) \leq n$.*

Proof. Notice r-dim$(X \setminus X_r) \leq n$. ∎

4. Dimension of a function

When trying to generalize Hurewicz Theorem from covering dimension to any other dimension theory, the issue arises of how to define the dimension of a function. Let us present an example of a Lipschitz function demonstrating that replacing $\sup\{\dim(f^{-1}(y)) \mid y \in Y\}$ by $\sup\{\mathrm{asdim}(f^{-1}(B)) \mid B \subset Y \text{ is bounded}\}$ does not work.

Proposition 4.1. *There is a Lipschitz function $f \colon X \to Y$ of metric spaces such that*

$$asdim(Y) = 0 = \sup\{asdim(f^{-1}(B)) \mid B \subset Y \text{ is bounded}\}$$

and $asdim(X) > 0$.

Proof. Let Y consist of points 2^n, $n \geq 1$, on the x-axis and X is the union of vertical segments I_n of length n and starting at 2^n. $f \colon X \to Y$ is the projection. Since $f^{-1}(B)$ is bounded for every bounded $B \subset Y$, $\sup\{\mathrm{asdim}(f^{-1}(B)) \mid B \subset Y \text{ is bounded}\} = 0$. Also, asdim$(Y) = 0$. However, asdim$(X) > 0$ as for each n it has n-components of arbitrarily large size. ∎

Remark 4.2. 4.1 shows the answer to a problem of Dranishnikov [14] in negative. That problem asks if l-asdim$(X) \leq$ l-asdim(Y)+l-asdimf^{-1}, where l-asdimf^{-1} is defined as $\sup\{$l-asdim$(f^{-1}(B)) \mid B \subset Y$ is bounded$\}$

and f is Lipschitz (see 2.2 for the definition of l-asdim). Notice l-asdim$(Y) = 0$, l-asdim$f^{-1} = 0$, and l-asdim$(X) \geq$ asdim$(X) > 0$ in 4.1.

In view of 4.1 we define asymptotic dimension of a function as follows:

Definition 4.3. Given a function $f\colon X \to Y$ of metric spaces we define the *asymptotic dimension* asdim(f) of f as the supremum of asymptotic dimensions of $A \subset X$ so that $f(A) \subset Y$ is of asymptotic dimension 0.

Definition 4.4. Given a function $f\colon X \to Y$ of metric spaces and given $m \geq 0$, an *m-dimensional control function* of f is a function $D_f\colon R_+ \times R_+ \to R_+$ such that for all $r_X > 0$ and $R_Y > 0$ any (∞, R_Y)-bounded subset A of X can be expressed as the union of $m + 1$ sets whose r_X-components are $D_f(r_X, R_Y)$-bounded.

Proposition 4.5. *Suppose $f\colon X \to Y$ is a function of metric spaces and $m \geq 0$. If asdim$(f) \leq m$, then f has an m-dimensional control function D_f.*

Proof. Fix $r_X > 0$ and $R_Y > 0$. Suppose for each n there is $y_n \in Y$ such that $A_n = f^{-1}(B(y_n, R_Y))$ cannot be expressed as the union of $m + 1$ sets whose r_X-components are n-bounded. The set $C = \bigcup\limits_{n=1}^{\infty} B(y_n, R_Y)$ cannot be bounded as asdim$(f^{-1}(C)) \leq m$ for any bounded subset C of Y. By passing to a subsequence we may arrange $y_n \to \infty$ and asdim$(C) = 0$, a contradiction. ∎

Definition 4.6. Given a function $f\colon X \to Y$ of metric spaces and given $k \geq m + 1 \geq 1$, an *(m, k)-dimensional control function* of f is a function $D_f\colon R_+ \times R_+ \to R_+$ such that for all $r_X > 0$ and $R_Y > 0$ any (∞, R_Y)-bounded subset A of X can be expressed as the union of k sets $\{A_i\}_{i=1}^{k}$ whose r_X-components are $D_f(r_X, R_Y)$-bounded so that any $x \in A$ belongs to at least $k - m$ elements of $\{A_i\}_{i=1}^{k}$.

Proposition 4.7. *Let $f\colon X \to Y$ be a function of metric spaces and $m \geq 0$. Suppose there is an m-dimensional control function $D_f^{(m+1)}\colon R_+ \times R_+ \to R_+$ of f. If one defines inductively functions $D_f^{(k)}$ for $k > m+1$ by $D_f^{(k)}(r_X, R_Y) = D_f^{(k-1)}(3r_X, R_Y) + 2r_X$, then each $D_f^{(k)}$ is an (m, k)-dimensional control function of f.*

Proof. The proof is by induction on k. The case of $k = m + 1$ is obvious. Suppose the result holds for some $k \geq n + 1$ and a subset A of X is (∞, R_Y)-bounded. There are k subsets $\{A_i\}_{i=1}^{k}$, each has $3r_X$-components bounded by $D_f^{(k)}(3r_X, R_Y)$ such that the union of any $m + 1$ of those sets covers A.

Define A_i' to be the r_X-neighborhood of A_i in A for $i \leq k$. Notice r_X-components of each A_i' are $(D_f^{(k)}(3r_X, R_Y) + 2r_X)$-bounded as they are contained in r_X-neighborhoods of $3r_X$-components of A_i.

Define A_{k+1}' as the union of all sets $\bigcap_{i \in S} A_i \setminus \bigcup_{i \notin S} A_i'$, where S is a subset of $\{1, \dots, k\}$ consisting of exactly $k - m$ elements.

Suppose $x \in A$ belongs exactly to $k - m$ sets A_i such that $i \leq k$ and let $S = \{i \leq k \mid x \in A_i\}$. If $x \notin A_{k+1}'$, then x must belong to A_j' for some $j \notin S$. Thus each $x \in A$ belongs to at least $k + 1 - m$ elements of $\{A_i'\}_{i=1}^{k+1}$.

Notice that any r_X-component of A_{k+1}' is contained in a single r_X-component of some A_j resulting in r_X-components of each A_i' being $(D_f^{(k)}(3r_X, R_Y) + 2r_X)$-bounded. ∎

Proposition 4.8. *Let $f \colon X \to Y$ be a function of metric spaces and $k, m \geq 0$. If $D_f \colon R_+ \times R_+ \to R_+$ is an (m, k)-dimensional control function of f, then for any $B \subset Y$ whose r_Y-components are R_Y-bounded, the set $f^{-1}(B)$ can be covered by k sets whose (r_X, r_Y)-components are $D_f(r_X, R_Y)$-bounded and every element of $f^{-1}(B)$ belongs to at least $k - m$ elements of that covering.*

Proof. Given an r_Y-component S of B express $f^{-1}(S)$ as $A_1^S \cup \dots \cup A_k^S$ such that r_X-components of A_i^S are $D_f(r_X, R_Y)$-bounded and every element of $f^{-1}(S)$ belongs to at least $k - m$ elements of that covering. Put $A_i = \bigcup_S A_i^S$ and notice each (r_X, r_Y)-component of A_i is contained in an r_X-component of some A_i^S. ∎

Theorem 4.9. *Let $k = m + n + 1$, where $m, n \geq 0$. Suppose $f \colon X \to Y$ is a large scale uniform function of metric spaces and $asdim(Y) \leq n$.*

$$asdim(X) \leq m + n$$

if there is an (m, k)-dimensional control function D_f of f. Moreover, if one can choose the coarseness control function c_f of f, the n-dimensional control function of Y, and D_f to be linear (respectively, a dilation), then X has a $(k - 1)$-dimensional control function that is linear (respectively, a dilation).

Proof. Let c_f be a coarseness control function of f. Let $D_Y \colon R_+ \to R_+$ be an (n, k)-dimensional control function of Y. Notice we may require $c_f(r) > r$, $D_Y(r) > r$, and $D_f(r, R) > r + R$ as we may redefine those functions by adding r or $r + R$ without losing their properties or type (linear, dilation or dilation near 0).

Given a number $r > 0$ we are going to construct a number $D_X(r)$ and represent the space X as a union of k sets $\{D^j\}_{j=1}^k$ with all r-components of D^j being $D_X(r)$-bounded for every j.

Define inductively a sequence of numbers $r_Y^{(n+1)} < R_Y^{(n+1)} < r_Y^{(n)} < R_Y^{(n)} < \cdots < r_Y^{(1)} < R_Y^{(1)} < r_Y^{(0)} < R_Y^{(0)}$ starting from $r_Y^{(n+1)} = c_f(r)$ and moving to lower indices so that for every i we have $R_Y^{(i)} = D_Y(r_Y^{(i)})$ and $r_Y^{(i)} = 3R_Y^{(i+1)}$.

Express Y as the union of k sets $\{A_i\}_{i=1}^k$ such that all $r_Y^{(0)}$-components of A_i are $R_Y^{(0)}$-bounded for every i. Since $A_i \subset Y$, we can express the set A_i as the union of k sets $\{U_i^j\}_{j=1}^k$ such that all $r_Y^{(i)}$-components of U_i^j are $R_Y^{(i)}$-bounded for every j and every point $y \in A_i$ belongs to at least m sets.

Define inductively a sequence of numbers $r_X^{(n+1)} < R_X^{(n+1)} < r_X^{(n)} < R_X^{(n)} \cdots < r_X^{(1)} < R_X^{(1)}$ starting with $r_X^{(n+1)} = r$ and for every i we have $R_X^{(i)} = D_f(r_X^{(i)}, R_Y^{(i)})$ and $r_X^{(i)} = 3R_X^{(i+1)}$.

For every i we express the set $f^{-1}(A_i)$ as the union of k sets $\{B_i^j\}_{j=1}^k$ such that all $(r_X^{(i)}, r_Y^{(i)})$-components of B_i^j are $(R_X^{(i)}, R_Y^{(i)})$-bounded for every j and every point $x \in f^{-1}(A_i)$ belongs to at least n sets.

Put $D_i^j = B_i^j \cap f^{-1}(U_i^j)$ and let D^j be the union of all D_i^j. Notice D^j's cover X by the use of Kolmogorov's argument: given $x \in X$ there is i so that $f(x) \in A_i$. The set of j's such that $x \in B_i^j$ has at least $k - m$ elements, the set of j's such that $f(x) \in U_i^j$ has at least $k - n$ elements, so they cannot be disjoint.

Notice all $(r_X^{(i)}, r_Y^{(i)})$-components of the set D_i^j are $(R_X^{(i)}, R_Y^{(i)})$-bounded. By 3.6 all $(r_X^{(n+1)}, r_Y^{(n+1)})$-components of the set D^j are $(3R_X^{(1)}, 3R_Y^{(1)})$-bounded. Since $(r_X^{(n+1)}, r_Y^{(n+1)}) = (r, c_f(r))$, by 3.3 all r-components of D^j are $3R_X^{(1)}$-bounded. Observe, $3R_X^{(1)}$ is a linear function of r (respectively, a dilation), if the coarseness control function c_f of f, the n-dimensional control function of Y, and D_f are linear (respectively, dilations). $\square$

Corollary 4.10. *Suppose $f: X \to Y$ is a function of metric spaces and $m \geq 0$. $asdim(f) \leq m$ if and only if f has an m-dimensional control function.*

Proof. In one direction use 4.5. In the other direction apply 4.9 in the case of $n = 0$. ∎

Theorem 4.11.

$$asdim(X) \leq asdim(f) + asdim(Y)$$

for any large scale uniform function $f\colon X \to Y$.

Proof. Follows from 4.9, 4.10, and 4.7. ∎

In [3] there is a concept of a family $\{X_\alpha\}$ of subsets of X satisfying $asdim(X_\alpha) \leq n$ uniformly. Notice that in our language it means there is one function that serves as an n-dimensional control function for all X_α.

Corollary 4.12 (Bell-Dranishnikov [3]). *Let $f\colon X \to Y$ be a Lipschitz function of metric spaces. Suppose that, for every $R > 0$,*

$$asdim\{f^{-1}(B_R(y))\} \leq n$$

uniformly (in $y \in Y$). If X is geodesic, then $asdim(X) \leq asdim(Y) + n$.

Proof. $asdim\{f^{-1}(B_R(y))\} \leq n$ uniformly means f has an n-dimensional control function, so apply 4.9 and 4.7. ∎

Remark 4.13. Notice Bell-Dranishnikov's version of Hurewicz type theorem 4.12 (see [3]) assumes that X is geodesic. We do not use this assumption in 4.11.

5. Asymptotic dimension of groups

J.Smith [27] showed that any two proper and left-invariant metrics on a given countable group G are coarsely equivalent. We generalize that result as follows.

Proposition 5.1. *Suppose $f\colon G \to H$ is a homomorphism of groups and d_G, d_H are left-invariant metrics on G and H, respectively. If $f\colon (G, d_G) \to (H, d_H)$ is coarsely proper (i.e., it sends bounded subsets of G to bounded subsets of H), then $f\colon (G, d_G) \to (H, d_H)$ is large scale uniform.*

Proof. Suppose $r > 0$. Since $f(B(1_G, r))$ is bounded, there is $R > 0$ such that $d_H(1_H, f(g)) < R$ for all $g \in B(1_G, r)$. If $x, y \in G$ satisfy $d_G(x, y) < r$, then $x^{-1} \cdot y \in B(1_G, r)$, so $d_H(f(x), f(y)) = d_H(1_H, f(x^{-1}y)) < R$. ∎

Corollary 5.2. *Suppose $f\colon G \to H$ is a homomorphism of groups and d_G, d_H are left-invariant metrics on G and H, respectively. If d_G is proper, then $f\colon (G, d_G) \to (H, d_H)$ is large scale uniform.*

Proof. Since bounded subsets of G are finite, f is coarsely proper. ∎

Corollary 5.3 (J.Smith [27]). *Any two proper left-invariant metrics on a group G are coarsely equivalent.*

The subsequent result is derived in [16] from the corresponding theorem for finitely generated groups. We deduce it directly from 4.11.

Theorem 5.4 (Dranishnikov-Smith [16]). *If $1 \to K \to G \to H \to 1$ is a short exact sequence of countable groups, then*

$$asdim(G) \leq asdim(K) + asdim(H).$$

Proof. It suffices to show $asdim(f) = k = asdim(K)$, where $f \colon G \to H$. Given $R_H > 0$, the subspace $f^{-1}(B(1_H, R_H))$ is in a bounded neighborhood of K as $B(1_H, R_H)$ is finite, so it has the same asymptotic dimension as K. Fix the dimension control function $D(r_G, R_H)$ for that space. Notice that all spaces $f^{-1}(B(h, R_H))$, $h \in H$, are isometric to $f^{-1}(B(1_H, R_H))$, so $D(r_G, R_H)$ is a dimension control function of all of them. By 4.10, $asdim(f) \leq k$ and by applying 4.11 we are done. ∎

In [16] the asymptotic dimension of a group G is defined as the supremum of $asdim(H)$, H ranging through all finitely generated subgroups of G. We provide an alternative definition which will be applied in the next section.

Definition 5.5. Given a finite subset S of a group G, an S-*component* of G is an equivalence class of G of the relation $x \sim y$ iff x can be connected to y by a finite chain x_i so that $x_i^{-1} \cdot x_{i+1} \in S$ for all i.

A family of subsets $\{A_i\}_{i \in J}$ is S-*bounded* if $x^{-1} \cdot y \in S$ for each $i \in J$ and all $x, y \in A_i$.

Proposition 5.6. *Let G be a group. $asdim(G) \leq n$ if and only if for each finite subset S of G there is a finite subset T of G and a decomposition $G = A_1 \cup \ldots \cup A_{n+1}$ such that S-components of A_i are T-bounded for all $i \leq n + 1$.*

Proof. Suppose for each finite subset S of G there is a finite subset T of G and a decomposition $G = A_1 \cup \ldots \cup A_{n+1}$ such that S-components of A_i are T-bounded for all $i \leq n + 1$. Given a countable subgroup H of G and given a proper left-invariant metric d_H on H, we put $S = B(1_H, r)$ and $R = \max\{d_H(1_H, t) \mid t \in T \cdot T \cap H\}$ (by $T \cdot T$ we mean all products $t \cdot s$, where $t, s \in T$). Notice that r-components of $A_i \cap H$ are R-bounded.

Conversely, suppose $asdim(H) \leq n$ for all finitely generated subgroups H of G. Given a finite subset S of G let H be the subgroup of G generated by S and let d be the word metric on H induced by S. Since $asdim(H) \leq n$, there is a decomposition of H into $A_1 \cup \ldots \cup A_{n+1}$ such that 1-components of A_i are m-bounded for all $i \leq n+1$ and some $m > 0$. Let $T = B(1_H, m + 1)$. Pick a representative $g_j \in G$, $j \in J$, of

each left coset of H in G. Put $B_i = \bigcup\limits_{j \in J} g_j \cdot A_i$ and notice S components of B_i are T-bounded. ∎

6. Linearly controlled asymptotic dimension of groups

Unlike the asymptotic dimension of countable groups, the asymptotic Assouad-Nagata dimension may depend on the left-invariant metric. Piotr Nowak [23] constructed finitely generated groups G_n of asymptotic dimension $n \geq 2$ and $\mathrm{asdim}_{AN}(G_n) = \infty$. Obviously, there is no such example for $n = 0$ as finitely generated groups of asymptotic dimension 0 are finite. However, 6.4 does provide a countable group G with a proper, left-invariant metric d such that $\mathrm{asdim}(G, d) = 0$ and $\mathrm{asdim}_{AN}(G, d) = \infty$.

The purpose of this section is to solve in negative the following two problems of Dranishnikov [14].

Problem 6.1. Does l-asdim$(X) = \mathrm{asdim}_{AN}(X)$ hold for metric spaces?

Problem 6.2. Find a metric space X of minimal $\mathrm{asdim}(X)$ such that $\mathrm{asdim}(X) < $ l-asdim(X).

Here is an answer to 6.2.

Proposition 6.3. *There is a proper, left-invariant metric d_G on $G = \bigoplus\limits_{n=2}^{\infty} \mathbb{Z}/n$ such that l-asdim$(G, d_G) > 0 = asdim(G, d_G)$.*

Proof. Pick a generator g_n of $\mathbb{Z}/n$ and assign it the norm of n for $n \geq 2$. Extend the norm over all elements g of G as the minimum of $\sum\limits_{n=1}^{\infty} |k_n| \cdot n$, where $g = \sum\limits_{n=1}^{\infty} k_n \cdot g_n$ and k_n are integers. Notice that $d_G(g, h) := |g - h|$ is a proper and invariant metric on G. Suppose there is an unbounded subset U of $\mathbb{R}_+$ such that for some $C > 0$ all r-components of G are $(C \cdot r)$-bounded for $r \in U$. Pick $r \in U$ such that $4C + 4 < r$ and let n be an integer such that $r - 2 < 2n \leq r$. Notice all $2n$-components of G are contained in some r-components of G, so they are $(n-1) \cdot (n+1)$-bounded as $C < \frac{r-4}{4} \leq \frac{n-2}{2}$ and $r < 2n + 2$. However, the $2n$-component of 0 contains $\mathbb{Z}/2n$ and $|n \cdot g_{2n}| = 2n^2 > (n-1) \cdot (n+1)$, a contradiction. ∎

Proposition 6.4. *There is an Abelian torsion group G with a proper invariant metric d_G such that l-asdim$(G, d_G) = 0$ and $asdim_{AN}(G, d_G) = \infty$.*

Proof. Consider $\mathbb{Z}^{n+1}$ with the standard word metric ρ_n. Since $\mathrm{asdim}(\mathbb{Z}^{n+1}) > n$, there is $r(n) > 0$ such that any decomposition of

$\mathbb{Z}^{n+1}$ as the union $X_0 \cup \ldots \cup X_n$ forces one X_i to have $r(n)$-components without a common upper bound. Without loss of generality we may assume $r(n)$ is an integer greater than n. Put $s(n) = 8 \cdot n \cdot r(n)$ and consider $G_n = (\mathbb{Z}_{s(n)})^{n+1}$ with the metric d_n equal to $t(n)$ times the standard word metric. Numbers $t(n)$ are chosen so that $t(n+1) > \mathrm{diam}(G_1) + \ldots + \mathrm{diam}(G_n)$ for all $n \geq 1$. Since $\mathrm{diam}(G_n) = 4nt(n)r(n)$, we want $t(n+1) = 1 + 4 \cdot t(1)r(1) + \ldots + 4n \cdot t(n)r(n)$ for all $n \geq 1$.

The group G is the direct sum of all G_n with the obvious metric d: $d(\{x_n\}, \{y_n\}) = \sum d_n(x_n, y_n)$. Notice that $\frac{t(n)}{2}$-components of G are of size at most $t(n)$. Indeed, any $\frac{t(n)}{2}$-component of G is contained in $x + G_1 \oplus \ldots \oplus G_{n-1}$ for some $x \in G$. Thus l-asdim$(G, d) = 0$.

Consider the projection $\pi_n \colon \mathbb{Z}^{n+1} \to G_n$ and notice it is $t(n)$-Lipschitz. Moreover, if two points x and y in $\mathbb{Z}^{n+1}$ satisfy $\rho_n(x, y) \leq 4nr(n)$, then $d_n(x, y) = t(n) \cdot \rho_n(x, y)$.

Suppose G_n equals $Y_0 \cup \ldots \cup Y_n$. Assume $r(n)$-components of some $X_j = \pi_n^{-1}(Y_j)$ are not uniformly bounded. Therefore there is a sequence $x_0, \ldots, x_k$ in X_j such that $\rho_n(x_i, x_{i+1}) < r(n)$ and $(n+2) \cdot r(n) > \rho_n(x_0, x_k) > n \cdot r(n)$. Assume such a sequence does not exist with number of elements smaller than $k + 1$. Therefore $d_n(\pi_n(x_i), \pi_n(x_{i+1})) = t(n) \cdot \rho_n(x_i, x_{i+1}) < t(n) \cdot r(n)$ and $(n+2) \cdot t(n)r(n) > \rho_n(\pi_n(x_0), \pi_n(x_k)) > n \cdot t(n)r(n)$. Thus, one $(t(n) \cdot r(n))$-component of Y_i is of diameter bigger than $n \cdot t(n)r(n)$.

Suppose asdim$_{AN}(G, d) = k < \infty$ and $D_G(r) = c \cdot r + b$ is a k-dimension control function of (G, d). Pick $n > \max(k, |c|, |b|) + 2$ and choose a decomposition $Y_0 \cup \ldots \cup Y_k$ of G so that $(t(n) \cdot r(n))$-components of each Y_i are bounded by $c \cdot t(n) \cdot r(n) + b < (c \cdot t(n) + 1) \cdot r(n)$. Notice there is a 1-Lipschitz projection $p_n \colon G \to G_n$. Therefore each set $p_n(Y_i)$ has $(t(n) \cdot r(n))$-components bounded by $(c \cdot t(n) + 1) \cdot r(n)$. On the other hand, for some j, there is an $(t(n) \cdot r(n))$-component of $p_n(Y_j)$ of diameter bigger than $nt(n) \cdot r(n)$. Thus, $c \cdot t(n) + 1 > nt(n)$ and $c > n - 1/t(n)$, a contradiction. ∎

Remark 6.5. Proposition 6.4 solves Problem 6.1 in negative.

Proposition 6.6. *For any countable group G there is a proper left-invariant metric d_G such that* asdim$_{AN}(G, d_G) = $ asdim(G).

Proof. Let asdim$(G) = n - 1$. Express G as the union of an increasing sequence $S_0 \subset S_1 \ldots$ of its finite subsets so that $\{1_G\} = S_0$, each S_i is symmetric, and for each i there is a decomposition of G as $A_1^i \cup \ldots \cup A_n^i$ so that S_i-components of A_k^i are S_{i+1}-bounded. Define $d_G(x, y)$ as the smallest i so that $x^{-1} \cdot y \in S_i$. Notice that in the metric d_G all i-components of A_k^i are $(i+1)$-bounded. ∎

7. Assouad-Nagata dimension

Definition 7.1. Given a function $f\colon X \to Y$ of metric spaces we define the *Assouad-Nagata dimension* $\dim_{AN}(f)$ of f as the minimum of m for which there is an m-dimensional control function $D_f(r_X, R_Y)$ of the form $a \cdot r_X + b \cdot R_Y$.

Theorem 7.2. *If $f\colon X \to Y$ is a Lipschitz function of metric spaces, then $dim_{AN}(X) \le dim_{AN}(f) + dim_{AN}(Y)$.*

Proof. Notice in the proof of 4.9 the resulting dimensional control function of X is a dilation if c_f, D_X, and D_f are dilations. ∎

Remark 7.3. Notice the proof of of 4.9 can be analyzed to give a version of 7.2 for microscopic Assouad-Nagata dimension.

8. Asymptotic Assouad-Nagata dimension

Definition 8.1. Given a function $f\colon X \to Y$ of metric spaces we define the *asymptotic Assouad-Nagata dimension* $\mathrm{asdim}_{AN}(f)$ of f as the minimum of m for which there is an m-dimensional control function $D_f(r_X, R_Y)$ of the form $a \cdot r_X + b \cdot R_Y + c$.

Theorem 8.2. *If $f\colon X \to Y$ is an asymptotically Lipschitz function of metric spaces, then $asdim_{AN}(X) \le asdim_{AN}(f) + asdim_{AN}(Y)$.*

Proof. Notice in the proof of 4.9 the resulting dimensional control function of X is linear if c_f, D_X, and D_f are linear. ∎

Proposition 8.3. *Let $n \ge 0$. If (G, d_G) is a group equipped with a proper, left-invariant metric d_G, then the following conditions are equivalent:*

    a. *$asdim_{AN}(G, d_G) \le n$.*
    b. *There are constants $C, M > 0$ such that the function $r \to M \cdot r + C$ is an n-dimensional control function for all finitely generated subgroups of G.*

Proof. a) $\implies$ b) follows from the proof of 3.10. b) $\implies$ a) is obvious. ∎

Proposition 8.4. *If $1 \to K \to G \to H \to 1$ is an exact sequence and G is a finitely generated group, then there are word metrics d_G on G and D_H on H such that $f\colon (G, d_G) \to (H, d_H)$ is 1-Lipschitz and for any m-dimensional control function D_K on K the function*

$$D_f(r_G, R_H) := D_K(r_G + 2R_H) + 2R_H$$

is an m-dimensional control function of f.

Proof. Let S be a symmetric finite set of generators for G. Let d_G be the word metric on G induced by S and let d_H be the word metric on H induced by $f(S)$.

Let $B = B(1_H, R_H)$ and $A = f^{-1}(B)$. Notice $A \subset B(K, R_H)$. Indeed, if $d_H(1_H, f(a)) < i = R_H$, then $f(a) = \prod_{i=1}^{m} f(s_i)$ such that $k < R_H$ and $s_i \in S$ for $i \leq m$. Consequently, $a = k \cdot x$ so that $k \in K$ and $x = \prod_{i=1}^{m} s_i$ has the property $d_G(1_G, x) < R_H$.

Now $d_G(k \cdot s, k) = d_G(s, 1_G) \leq i - 1 < i$ and $a \in B(K, R_H)$. By 3.7, the function $D_f(r_G, R_H) := D_K(r_G + 2R_H) + 2R_H$ is an m-dimensional control function of $B(K, R_H)$. Since all $f^{-1}(B(y, R_H))$ are isometric to A, we are done. ∎

Corollary 8.5. *If $1 \to K \to G \to H \to 1$ is an exact sequence of groups so that G is finitely generated, then*

$$\mathrm{asdim}_{AN}(G, d_G) \leq \mathrm{asdim}_{AN}(K, d_G|K) + \mathrm{asdim}_{AN}(H, d_H)$$

for any word metrics metrics d_G on G and d_H on H.

Proof. Use the metrics as in 8.4. That way

$$\mathrm{asdim}_{AN}(f) \leq \mathrm{asdim}_{AN}(K, d_G|K),$$

so applying 8.2 one gets the desired inequality. ∎

One is tempted to define the linear asymptotic dimension of arbitrary groups as the supremum of $\mathrm{asdim}_{AN}(H)$ for all finitely generated subgroups H of G, However, one runs into problems with that definition.

Question 8.6. Suppose G is a finitely generated group and H is its finitely generated subgroup. Does $\mathrm{asdim}_{AN}(H) \leq \mathrm{asdim}_{AN}(G)$ hold? Is there a case of $\mathrm{asdim}_{AN}(H) = \infty$ and $\mathrm{asdim}_{AN}(G) < \infty$?

Question 8.7. Suppose G is a finitely generated group and H is its finitely generated subgroup. Does $\mathrm{asdim}_{AN}(H) = \mathrm{asdim}_{AN}(H, d_G|H)$ hold for any word metric d_G on G?

Question 8.8. Suppose G and H are two finitely generated groups of finite asymptotic Assouad-Nagata dimension. Is $\mathrm{asdim}_{AN}(G*H)$ finite? Does $\mathrm{asdim}_{AN}(G * H) = \max(\mathrm{asdim}_{AN}(G), \mathrm{asdim}_{AN}(H), 1)$ hold?

To complete this section we state and prove a version of the the Hurewicz theorem for groups acting on spaces of finite asymptotic Assouad-Nagata dimension. For this we need the concept of R-stabilizers. Let G act on the metric space X by isometries and let $R > 0$. Given

$x_0 \in X$ the R-*stabilizer* of x_0 is defined by $W_R(x_0) = \{\gamma \in G : d(\gamma \cdot x_0, x_0) \leq R\}$.

Theorem 8.9. *Let G be a finitely generated group acting by isometries on a metric space X of finite asymptotic Assouad-Nagata dimension. Fix a point $x_0 \in X$. If there are constants $m, b, c > 0$ such that $r \to m \cdot r + b \cdot R + c$ is a k-dimensional control function of $W_R(x_0)$ for all R, then*

$$asdim_{AN}(G) \leq k + asdim_{AN}(X).$$

Proof. Let S be a symmetric generating set for G and let $\lambda = \max\{d_X(s \cdot x_0, x_0) | s \in S\}$. Define $\pi \colon G \to X$ by $\pi(\gamma) = \gamma \cdot x_0$ and notice that π is λ-Lipschitz. Also, $\pi^{-1}(B_R(g \cdot x_0)) = gW_R(x_0)$ shows that the sets $(\pi^{-1}(B_R(g \cdot x_0)))$ are all isometric to $W_R(x_0)$ and we can use 8.2 to get our result. ∎

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